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Maltsev categories and Maltsev operations

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Abstract

We define and investigate Maltsev operations on an arbitrary base category, and study their relationship to the notion of Maltsev category.

1. Introduction

The basic property of a Maltsev category $\mathscr{E}$ is that any reflexive relation in $\mathscr{E}$ is already an equivalence relation; if the category is regular, so that we can consider composition of equivalences, then Maltsev categories are exactly "commutative congruence categories".

It is well known that, for varieties, this good behaviour of relations corresponds to the existence of a suitable ternary operation [7]; in the case of groups, for example, such an operation $p: G^3 \to G$ is defined by $p(x, y, z) = x - y + z$.

However, there are many examples of non-varietal Maltsev categories, like topological and localic groups [6], and in general models of Maltsev varieties in good categories, or dual categories of topoi and pretopoi (see [2] for this last example).

Our aim is to show that the three cases: varieties, topological or localic groups and dual categories of topoi admit the same interpretation of being Maltsev, i.e. they are monadic categories equipped with a Maltsev operation on the base category. To do this, we introduce and investigate the notion of U-Maltsev operation on a category $\mathscr{E}$, where $U: \mathscr{E} \to \mathscr{S}$ is a lex functor. If $U = Id: \mathscr{E} \to \mathscr{E}$, we get Johnstone's definition of a naturally Maltsev category [5].

Furthermore, in the case $\mathscr{S} = SET$, by using Richter's results [9], we show that there is an interesting correspondence between monadic Maltsev categories and Maltsev operations with respect to the forgetful functor.

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2. Maltsev operations

We recall the basic definitions and results concerning Maltsev categories; for further details see [2–4].

In the following $\mathscr{E}$ will always denote a lex category (= left exact category, i.e. a category with finite limits).

Definition 1. A category $\mathscr{E}$ is a Maltsev category if and only if it satisfies the following axiom (M):

(M) for any object X in $\mathscr{E}$, any reflexive relation $R \rightarrowtail X \times X$ is an equivalence relation.

We will need some basic properties of Maltsev categories.

Proposition 2. *For a category $\mathscr{E}$, the following facts are equivalent:*

(1) *axiom* (M);

(2) *for any object X in $\mathscr{E}$, any reflexive relation on X is symmetric;*

(3) *for arbitrary objects X and Y in $\mathscr{E}$, any relation $R \rightarrowtail X \times Y$ is difunctional* [4].

Proposition 3. *If $\mathscr{E}$ is a regular category, the following facts are equivalent:*

(1) *axiom* (M);

(2) *composition of equivalence relations is commutative;*

(3) *the composite of two equivalence relations is an equivalence relation.*

Let us suppose now that $\mathscr{E}$ admits a lex (= left exact; i.e. finite limits preserving) functor $U: \mathscr{E} \to \mathscr{S}$ with $\mathscr{S}$ a lex category.

We will denote by U^n the n-fold product of U by itself, by π_i the corresponding projections and by $\Delta: U \to U^2$ the diagonal.

Definition 4. Let $U: \mathscr{E} \to \mathscr{S}$ be a lex functor between lex categories. By a U-Maltsev operation p on $\mathscr{E}$ is meant a natural transformation $p: U^3 \to U$ satisfying the following two axioms:

(1) $p \cdot (1 \times \Delta) = \pi_1$,

(2) $p \cdot (\Delta \times 1) = \pi_2$.

Observe that, by definition, a U-Maltsev operation on $\mathscr{E}$ determines a structure of internal Maltsev algebra on U in the "category" of functors from $\mathscr{E}$ to $\mathscr{S}$.

Remark 5. If $U = Id: \mathscr{E} \to \mathscr{E}$, Definition 4 gives Johnstone's notion of naturally Maltsev category [6].

Proposition 6. *Let $U: \mathscr{E} \to \mathscr{S}$ be a lex functor between lex categories. If U is conservative (= isomorphism reflecting) and $\mathscr{E}$ admits a U-Maltsev operation p, then $\mathscr{E}$ is a Maltsev category.*

Proof. By Proposition 2, it suffices to show that any reflexive relation $((a,b): R \rightrightarrows X, \rho: X \to R)$ in $\mathscr{E}$ is symmetric.

Consider the corresponding relation $(Ua, Ub): UR \rightrightarrows UX$ in $\mathscr{S}$; UR is clearly reflexive; hence, by applying p, we can define a morphism $\sigma: UR \to UR$ as the following composition:

$$p_R \cdot \langle U\rho \cdot Ua, 1, U\rho \cdot Ub \rangle : UR \to UR^3 \to UR.$$

It is easy to verify that σ is a symmetry on UR.

To conclude the proof we will need the following lemma:

Lemma 7. *A lex functor $U: \mathscr{E} \to \mathscr{S}$ is conservative if and only if, for any Y in $\mathscr{E}$ and for any pair of subobjects $(m: R \rightarrowtail Y, n: S \rightarrowtail Y)$ of Y such that there exists an isomorphism $\xi: UR \to US$ between the induced subobjects, there exists an isomorphism $\phi: R \to S$ with $n \cdot \phi = m$ and $U(\phi) = \xi$.*

Proof of Proposition 6 (*Conclusion*). Consider now the subobjects $\langle a, b \rangle : R \rightarrowtail X \times X$ and $\langle b, a \rangle : R \rightarrowtail X \times X$. Since $\sigma: UR \to UR$ is an isomorphism between the induced subobjects in $\mathscr{S}$, by Lemma 7 the lifting $\phi: R \to R$ of σ is a symmetry for R in $\mathscr{E}$. $\square$

Corollary 8. *If $\mathscr{E}$ is naturally Maltsev, then $\mathscr{E}$ is a Maltsev category.*

Proof. Trivial by Remark 5. $\square$

Corollary 9. *If $U: \mathscr{E} \to \mathscr{S}$ is a monadic functor, $\mathscr{S}$ is lex, and $p: U^3 \to U$ is a U-Maltsev operation on $\mathscr{E}$, then $\mathscr{E}$ is a Maltsev category.*

Proof. Trivial by Proposition 6 since in this case U is conservative. $\square$

Basic examples of the monadic case can be obtained by the following choices of p. Let $\mathscr{E} = GRP$ be the category of groups, then $p_G: (UG)^3 \to UG$ is defined as $p_G(x, y, z) = x - y + z$, for any group G. Let now $\mathscr{E} = HEYT$ be the category of Heyting algebras, then $p_H: (UH)^3 \to UH$ is defined by

$$p_H(x, y, z) = (y \to (x \wedge z)) \wedge (x \vee z)$$

for any Heyting algebra H.

If the Maltsev category $\mathscr{E}$ is a variety as in the above examples, then it is called a Maltsev variety.

Corollary 10. *For any lex category $\mathscr{S}$, the corresponding category $\mathscr{E} = GRP(\mathscr{S})$ of internal groups is a Maltsev category.*

Proof. Apply Proposition 6 to the forgetful functor $U: GRP(\mathscr{S}) \to \mathscr{S}$, by defining the U-operation as in the case of SET.

Clearly Corollary 10 applies to models in $\mathscr{S}$ of any Maltsev variety.

Corollary 11. *If $\mathscr{E}$ is a topos, then $\mathscr{E}^o$ is a Maltsev category.*

Proof. Let $\mathscr{E}$ be a topos and Ω denote the subobjects classifier of $\mathscr{E}$, then $\mathscr{E}^o$ is monadic over $\mathscr{E}$ with $U:\mathscr{E}^o \to \mathscr{E}$ defined by $U = \Omega^-$ [8]. For any X in $\mathscr{E}$, the object $UX = \Omega^X$ is an internal Heyting algebra in $\mathscr{E}$, hence it admits an internal natural Maltsev operation $p_X:(\Omega^X)^3 \to \Omega^X$. □

Notice that *GRP*, *HEYT* and $\mathscr{E}^o$, with $\mathscr{E}$ a topos, are examples of categories equipped with a U-Maltsev operation that are not naturally Maltsev.

In the case of a monadic category $\mathscr{E}$ over $\mathscr{S} = SET$, the converse of Proposition 6, forcing the existence of a U-operation whenever $\mathscr{E}$ is Maltsev, holds:

Theorem 12. *Let $U:\mathscr{E} \to SET$ be a monadic functor, then the following are equivalent:*

(1) *$\mathscr{E}$ is a Maltsev category;*
(2) *there exists a U-Maltsev operation on $\mathscr{E}$;*
(3) *the induced triple $T:SET \to SET$ is an internal Maltsev algebra in the "category" of endofunctors on SET.*

Proof. (2) implies (1) follows by Proposition 6. (2) implies (3) is trivial, since $T:SET \to SET$ is the composite $T = U \cdot F$ where $F:SET \to \mathscr{E}$ denotes the left adjoint to U.

To show (3) implies (2), denote by $q_S:(TS)^3 \to TS$ the natural Maltsev operation on TS, with S in SET. Then, for any T-algebra X in $\mathscr{E}$ with $X = (S, h: TS \to S)$ and $UX = S$, consider its canonical presentation as split sequence:

$$TTS \rightrightarrows TS \to S;$$

hence, by $q_{TS}:(TTS)^3 \to TTS$ and by $q_S:(TS)^3 \to TS$, we get an induced morphism $p_X:(UX)^3 = S^3 \to UX = S$. It is easy to verify that p is natural and satisfies Definition 4.

In remains to prove that (1) implies (2). To do that it suffices to notice that $U \simeq hom(F(1), -)$ with $F(1)$ projective in $\mathscr{E}$ and apply Richter's construction [9] to obtain a Maltsev co-operation $t: F(1) \to F(1) + F(1) + F(1)$. Clearly t induces a natural Maltsev operation defined by

$$p_X = hom(t, X): hom(F(1), X)^3 \simeq hom(F(1) + F(1) + F(1), X) \to hom(F(1), X).$$

□

We conclude with a remark that will allow us to construct new examples of Maltsev operations from the previous ones. Recall that for any lex category $\mathscr{E}$, it is possible to define, in a functorial way, a corresponding free (Barr) exact category $\mathscr{E}_{ex}$. The objects of $\mathscr{E}_{ex}$ are pseudo equivalence relations (R, X), i.e., internal graphs $(d_0^R, d_1^R): R \rightrightarrows X$ in $\mathscr{E}$ (not necessarily jointly monic) which are reflexive, symmetric and transitive. An arrow $[f]:(R, X) \to (S, Y)$ in $\mathscr{E}_{ex}$, is an equivalence class of an arrow $f: X \to Y$

in $\mathscr{E}$ which is "compatible with the relations on X and Y", in the sense that there exists an arrow $f_1 : R \to S$ such that $d_0^S \cdot f_1 = f \cdot d_0^R$ and $d_1^S \cdot f_1 = f \cdot d_1^R$. Two such arrows $[f]$ and $[g]$ are said to be equivalent if there exists $\Sigma : X \to S$ with $d_0 \cdot \Sigma = f$ and $d_1 \cdot \Sigma = g$.

Now, let $U : \mathscr{E} \to \mathscr{S}$ be a lex functor between lex categories, and denote by $U_{ex} : \mathscr{E}_{ex} \to \mathscr{S}_{ex}$ the induced exact functor.

If p is a U-Maltsev operation on $\mathscr{E}$ then since for any element (R, X) of $\mathscr{E}_{ex}$ it holds $U_{ex}(R, X) = (UR, UX)$, we can define a U_{ex}-Maltsev operation p_{ex} on $\mathscr{E}_{ex}$ by $(p_{ex})_{(R,X)} = (p_R, p_X)$.

Let $\mathscr{E} = GRP$ be the category of groups and consider the canonical forgetful functor U and Maltsev operation p; since $SET_{ex} \simeq SET$, we get an exact functor $U_{ex} : GRP_{ex} \to SET$ equipped with an induced U_{ex}-Maltsev operation $p_{ex} : U_{ex}^3 \to U_{ex}$ on GRP_{ex}. However GRP_{ex} does not seem to be a Maltsev category.

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